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A Chebyshev Expansion of Singular Integrodifferential Equations with a $\partial^2 \ln|s-t|/\partial s\, \partial t$ Kernel

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Singular integrodifferential equations of the first kind with a symmetric kernel having a $\partial^2 \ln|s-t|/\partial s\, \partial t$ singularity (i.e., a derivative of a Cauchy-type kernel) are studied. An algorithm based on the expansion in Chebyshev polynomials of the second kind is presented. The method is applied to two-dimensional wave-scattering, and convergence is demonstrated.

1. INTRODUCTION

Two-dimensional wave-scattering by thin soft scatterers lead to integrodifferential equation of the form

$$\int_{-1}^{1} F(t) \left\{ \left[\frac{\partial^2}{\partial s\, \partial t} + B(s,t) \right] \ln|s-t| + C(s,t) \right\} dt = g(s), \qquad |s| < 1, \qquad (1.1)$$

where $F(t)$ is the unknown function, $g(s)$ is the forcing function, and $B(s,t)$ and $C(s,t)$ are symmetric regular kernels [1]. By the notation $\partial^2 \ln|s-t|/\partial s\, \partial t$ we mean

$$\int_{-1}^{1} F(t) \frac{\partial^2}{\partial s\, \partial t} \ln|s-t|\, dt \equiv \frac{\partial}{\partial s} \int_{-1}^{1} F(t) \frac{dt}{t-s}. \qquad (1.2)$$

Integrodifferential equations of the Cauchy type are generally treated within the framework of Cauchy-type singular equations [2]. For the special case of (1.1) the transformation is accomplished by integrating both sides of (1.1) over s, and using specified boundary conditions of $F(t)$.

However, by doing so we lose the symmetry of (1.1). In this paper we present a direct solution of Eq. (1.1), which preserves the symmetry of the equation.

It is well known that for smooth enough regular kernels and forcing function, the solution $F(t)$ exhibits square-root zeros at the edges of the scatterer [5]. We rewrite Eq. (1.1) in the form

$$\int_{-1}^{1} (1-t^2)^{1/2} f(t) \left\{ \left[\frac{\partial^2}{\partial s\, \partial t} + B(s,t) \right] \ln|s-t| + C(s,t) \right\} dt = g(s), \qquad |s| < 1. \qquad (1.3)$$

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Our approach for solving (1.3) is similar to the one presented in an accompanying article on singular integral equations with a logarithmic kernel [3]. There, Chebyshev polynomials of the first kind are eigenfunctions of the operator; here, Chebyshev polynomials of the second kind play the same role since

$$\int_{-1}^{1} (1-t^2)^{1/2}\, U_n(t) \frac{\partial^2}{\partial s\, \partial t} \ln |s-t|\, dt = -\pi(n+1)\, U_n(s), \qquad |s|<1. \tag{1.4}$$

The application of (1.4) for the solution of (1.3) with constant regular kernels has been described by Butler and Wilton [4]. This special case of (1.3) appears in the quasistatic limit of scattering by a strip.

In this article we generalize the results of Butler and Wilton, by expanding all relevant functions in Chebyshev polynomials of the second kind. By using functional analysis we study the properties of the solution and its convergence. We further describe the numerical algorithm, and finally, present several numerical results.

2. Theory

As pointed out, Eq. (1.4) naturally suggests that the solution of (1.3) be accomplished by expansion in Chebyshev polynomials of the second kind. Let $L^2(\Gamma, \rho)$ denote the Hilbert space of all complex-valued functions square integrable on $\Gamma = (-1, 1)$ with respect to the weight $\rho(t) = (1-t^2)^{1/2}$. It is well known that the Chebyshev polynomials of the second kind $\{U_n(x)\}$ comprise a complete orthogonal set in $L^2(\Gamma, \rho)$. Moreover, the series $\sum_{n=0}^{\infty} f_n U_n(x)$ with

$$f_n = \frac{2}{\pi} \int_{-1}^{1} (1-x^2)^{1/2}\, U_n(x) f(x)\, dx, \tag{2.1}$$

converges in the mean to the function $f(x)$. The norm in this Hilbert space is simply

$$\|f\|^2 = \int_{-1}^{1} (1-x^2)^{1/2}\, |f(x)|^2\, dx = \frac{\pi}{2} \sum_{n=0}^{\infty} |f_n|^2. \tag{2.2}$$

We study bounded integral operators $\mathscr{K}$ within $L^2(\Gamma, \rho)$ of the form

$$\mathscr{K} f = \int_{-1}^{1} (1-t^2)^{1/2}\, K(s,t) f(t)\, dt. \tag{2.3}$$

We start with completely continuous operators which obey the sufficient condition

$$\|\mathscr{K}\|^2 = \int_{-1}^{1} \int_{-1}^{1} |K(s,t)|^2\, (1-t^2)^{1/2}\, (1-s^2)^{1/2}\, ds\, dt < \infty. \tag{2.4}$$

We may prove

THEOREM I. *If $\mathscr{K}$ is a compact completely continuous operator, then the double expansion*

$$K_{MN}(s,t)=\sum_{m=0}^{M}\sum_{n=0}^{N}K_{mn}U_m(s)\,U_n(t), \tag{2.5}$$

with

$$K_{mn}=\frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}(1-s^2)^{1/2}\,(1-t^2)^{1/2}\,K(s,t)\,U_m(s)\,U_n(t)\,ds\,dt \tag{2.6}$$

converges in the mean to $K(s,t)$ as $M,N\to\infty$. That is, for any $\varepsilon>0$ there are M_0 and N_0 such that for every $M>M_0$ and $N>N_0$.

$$\int_{-1}^{1}\int_{-1}^{1}(1-s^2)^{1/2}\,(1-t^2)^{1/2}\,|K(s,t)-K_{MN}(s,t)|^2\,ds\,dt<\varepsilon. \tag{2.7}$$

Also,

$$\|\mathscr{K}\|^2=\frac{\pi^2}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|K_{mn}|^2. \tag{2.8}$$

We now prove Theorem II about the logarithmic kernel.

THEOREM II. *The integral operator $\mathscr{L}$*

$$\mathscr{L}f=\int_{-1}^{1}(1-t^2)^{1/2}\ln|s-t|\,f(t)\,dt, \tag{2.9}$$

where $f(t)\in L^2(\Gamma,\rho)$ is a self-adjoint completely continuous operator.

Proof. It is well known that

$$\int_{-1}^{1}\frac{\ln|s-t|\,T_n(t)}{(1-t^2)^{1/2}}\,dt=-\pi v_n T_n(s), \tag{2.10}$$

where $v_0=\ln 2$, $v_n=1/n$ $(n>0)$, and $T_n(t)$ are the Chebyshev polynomials of the first kind. We show that the operator $\mathscr{L}$ obeys condition (2.4). Using (2.10) we find

$$\begin{aligned}L_{mn}&=\frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}(1-s^2)^{1/2}\,(1-t^2)^{1/2}\,U_m(s)\,U_n(t)\ln|s-t|\,ds\,dt\\&=-\ln 2\delta_{m0}\delta_{n0}-\frac{8}{\pi^2}\sum_{l=1}^{\infty}\frac{1}{l}d_{ml}d_{nl},\end{aligned} \tag{2.11}$$

where

$$d_{ml}=\int_{-1}^{1}(1-s^2)^{1/2}\,U_m(s)\,T_l(s)\,ds=\frac{\pi}{2}\{\delta_{ml}-\delta_{m+2,l}\}. \tag{2.12}$$

Finally, we have,

$$L_{mn} = -\left\{\ln 2\delta_{m0}\delta_{n0} + \frac{2\delta_{mn}}{m}(1-\delta_{m0}) + \frac{2\delta_{mn}}{m+2} - \frac{2}{m}\delta_{m,n+2} - \frac{2}{n}\delta_{m+2,n}\right\}.$$

A direct calculation confirms that $\sum_{m,n=0}^{\infty} |L_{mn}|^2$ converges.

We further discuss the inverse operator of a completely continuous operator $\mathscr{K}$. Let $\mathscr{J}$ be bounded on a subspace $M \subseteq L^2(\Gamma, \rho)$, such that $\mathscr{J}\mathscr{K}$ and $\mathscr{K}\mathscr{J}$ are the identity operators for properly defined functions. If $\mathscr{K}$ is represented by a matrix K, and the bounded opertor $\mathscr{J}$ by the matrix J, then

$$J_{mn} = \frac{4}{\pi^2}(K^{-1})_{mn}. \tag{2.13}$$

The factor $4/\pi^2$ results from the fact that the Chebyshev polynomials, $\{U_n(x)\}$ are not normalized.

We now prove

THEOREM III. *The integrodifferential operator whose kernel is* $J(s,t) = \partial^2 \ln|s-t|/\partial s\,\partial t$ *is the inverse of the completely continuous operator* $\mathscr{K}$ *whose kernel is*

$$K(s,t) \sim -\frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(n+1)}U_n(s)\,U_n(t). \tag{2.14}$$

Proof. Obviously $\mathscr{K}$ is completely continuous since

$$\|\mathscr{K}\|^2 = \frac{\pi^2}{4}\left(-\frac{2}{\pi^2}\right)^2 \sum_{n=0}^{\infty}\frac{1}{(n+1)^2} = \frac{1}{6}. \tag{2.15}$$

It is well known that

$$\int_{-1}^{1}(1-t^2)^{1/2}\frac{U_n(t)}{t-s}\,dt = -\pi T_{n+1}(s). \tag{2.16}$$

Therefore,

$$\frac{\partial}{\partial s}\int_{-1}^{1}(1-t^2)^{1/2}\frac{U_n(t)}{t-s}\,dt = -\pi(n+1)\,U_n(s) \tag{2.17}$$

which is identical to (1.4). Thus

$$\begin{aligned} J_{mn} &= \frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}(1-t^2)^{1/2}\frac{\partial^2 \ln|s-t|}{\partial s\,\partial t}U_m(s)\,U_n(t)\,ds\,dt \\ &= (-2)(n+1)\,\delta_{mn}. \end{aligned} \tag{2.18}$$

However, this is identical to $(4/\pi^2)(K^{-1})_{mn}$, which proves our statement. Moreover,

$$\mathscr{T}f = \int_{-1}^{1} (1-t^2)^{1/2} J(s,t) f(t)\, dt \sim -\pi \sum_{n=0}^{\infty} (n+1) f_n U_n(s) \tag{2.19}$$

so that $\mathscr{T}f \in L^2(\Gamma, \rho)$ if and only if $\sum_{n=0}^{\infty} (n+1)^2 |f_n|^2 < \infty$. The last condition defines the subspace of $L^2(\Gamma, \rho)$ in which the operator $\mathscr{T}$ is defined.

We generalize Theorem III in the following form:

THEOREM IV. *Let $\mathscr{K}$ be a completely continuous operator, that obeys* (2.4), *and let $\mathscr{T}$ be its inverse. Let also $\mathscr{H}$ be a bounded operator. Then $\mathscr{T} + \mathscr{H}$ is the inverse of the operator $(\mathscr{I} + \mathscr{K}\mathscr{H})^{-1}\mathscr{K}$.*

Proof. If $\mathscr{K}$ is a completely continuous of the Hilbert–Schmidt type and $\mathscr{H}$ is bounded, then $\mathscr{K}\mathscr{H}$ obeys (2.4) also, and $(\mathscr{I} + \mathscr{K}\mathscr{H})^{-1}$ exists. However, $\mathscr{T} + \mathscr{H} = \mathscr{T}(\mathscr{I} + \mathscr{K}\mathscr{H})$ so that

$$(\mathscr{T} + \mathscr{H})^{-1} = (\mathscr{I} + \mathscr{K}\mathscr{H})^{-1}\mathscr{K}.$$

We conclude by showing that Eq. (1.3) obeys Theorem IV, and therefore can be solved by inversion. It is clear that the integral operator whose kernel is $B(s,t) \ln|s-t| + C(s,t)$ is bounded. The operator whose kernel is $\partial^2 \ln|s-t|/\partial s\, \partial t$ fulfills the requirements of the operator $\mathscr{T}$, so that the diagonal operator $\mathscr{K}$ is defined by (2.14) (Theorem III).

3. ALGORITHM

As described in Section 2, the solution of (1.3) is accomplished by expansion in Chebyshev polynomials of the second kind. The functions $g(s)$ and $f(t)$ are expanded by (2.1). The coefficients $\{g_n\}$ are calculated by a modified Chebyshev quadrature

$$g_n \simeq \frac{1}{M+1} \sum_{i=0}^{M} g(x_i)[T_m(x_i) - T_{m+2})], \qquad x_i = \cos\left[\frac{(2i+1)\pi}{2M+2}\right]. \tag{3.1}$$

The kernel $C(s,t)$ is expanded similarly

$$C(s,t) = \sum_{m,n=0}^{N} C_{mn} U_m(s)\, U_n(t), \tag{3.2}$$

where

$$C_{mn} \simeq \frac{1}{(M+1)^2} \sum_{i,j=0}^{M} C(x_i, x_j)[T_m(x_i) - T_{m+2}(x_i)][T_n(x_j) - T_{n+2}(x_j)]. \tag{3.3}$$

The matrix elements of $\partial^2 \ln|s-t|/\partial s\,\partial t$ have been calculated already (2.18). We are finally left with the matrix elements of $B(s,t)\ln|s-t|$. We use the formula

$$\ln|s-t| \sim -\sum_{l=0}^{\infty} v_l T_l(t)\, T_l(s), \qquad v_0 = \ln 2, \quad v_l = 2/l, \tag{3.4}$$

where $T_l(t)$ are the Chebyshev polynomials of the first kind, and find

$$\left(\frac{2}{\pi}\right)^2 \int_{-1}^{1}\int_{-1}^{1} (1-t^2)^{1/2}(1-s^2)^{1/2}\, U_m(s)\, U_n(t)\, B(s,t) \ln|s-t|\, ds\, dt$$

$$= \frac{1}{4}\sum_{l=0}^{\infty} \{B_{m-l,n-l} + B_{m+l,n-l} - B_{l-m-2,n-l} + B_{m-l,n+l} + B_{m+l,n+l}$$

$$- B_{l-m-2,n+l} - B_{m-l,l-n-2} - B_{m+l,l-n-2} + B_{l-m-2,l-n-2}\} \equiv \beta_{mn}. \tag{3.5}$$

The matrix equation for finding **f** is $f = Z^{-1}\mathbf{g}$, where

$$Z_{mn} = (\pi/2)\, C_{mn} - \pi\delta_{mn}(n+1) + (\pi/8)\,\beta_{mn}. \tag{3.6}$$

4. Numerical Examples

In this section we present numerical results for some integrodifferential equations corresponding to scattering of an H-polarized EM wave by an open thin two-dimensional scatterer. Let the scatterer be defined by the parametric equations $(x(t), y(t))$. Then the equation for the current density is

$$\int_{-1}^{1} f(t)(1-t^2)^{1/2} \left\{k^2\left(\frac{dx}{dt}\frac{dx}{ds} + \frac{dy}{dt}\frac{dy}{ds}\right) - \frac{\partial^2}{\partial s\,\partial t}\right\}$$

$$\times H_0^{(2)}(k\,|\boldsymbol{\rho}(s) - \boldsymbol{\rho}(t)|)\, dt = g(s). \tag{4.1}$$

We reduce (4.1) into the form (1.3) ([1]), and use the algorithm described.

The first example is the scattering of a plane wave by a strip $\boldsymbol{\rho}(t) = (wt, 0)$. The equation we have to solve is

$$\int_{-1}^{1} (1-t^2)^{1/2} f(t) \left\{k^2 w^2 - \frac{\partial^2}{\partial s\,\partial t}\right\} H_0^{(2)}(kw\,|s-t|)\, dt = kw \cos\alpha\, e^{jwkt\cos\alpha}. \tag{4.2}$$

The parameters for this example are $\alpha = 0$ (i.e., symmetrical excitation) and $kw = 3\pi$. The calculated coefficients $\{f_n\}$ are presented in Table I for different dimensions N of the matrix Z (Eq. (3.6)). The convergence of the results is obvious.

TABLE I

Chebyshev Expansion Coefficients $\{f_n\}$ for Plane-Wave Scattering by a Strip ($kw = 3\pi$)

n	$N = 6$	$N = 10$	$N = 14$
0	$0.63883 + j.02071$	$0.62896 + j.02096$	$0.62891 + j.02100$
2	$0.19363 + j.07175$	$0.16439 + j.07705$	$0.16425 + j.07721$
4	$0.01842 + j.09014$	$-0.01575 + j.11260$	$-0.01585 + j.11286$
6	$-0.12693 + j.03784$	$-0.11134 + j.06874$	$-0.11116 + j.06883$
8		$0.05760 - j.07372$	$0.05762 - j.07411$
10		$-0.01420 + j.02441$	$-0.01454 + j.02522$
12			$0.00240 - j.00503$
14			$-0.00029 + j.00068$

The second example is the scattering of a plane wave by a semicircular cylinder $\boldsymbol{\rho}(s) = (a\cos(\pi t/2, a\sin(\pi t/2))$. The equation for the current reads

$$\int_{-1}^{1} (1-s^2)^{1/2} f(s) \left\{ k^2a^2 \cos\left(\frac{\pi}{2}(s-t)\right) - \frac{\partial^2}{\partial s\,\partial t} \right\} H_0^{(2)}\left[2ka \sin\left(\frac{\pi}{4}|s-t|\right)\right] ds$$

$$= -ka\,\frac{\pi}{2}\cos\left(\alpha - \frac{\pi t}{2}\right) e^{jka\cos\left(\alpha - \frac{\pi t}{2}\right)}. \tag{4.3}$$

The results for $\alpha = \pi/2$ and $ka = \pi$ are presented in Table II. Again a rather fast convergence is observed.

TABLE II

Chebyshev Expansion Coefficients $\{f_n\}$ for Plane-Wave Scattering by a Semicircular Strip ($ka = \pi$)

n	$N = 6$	$N = 10$	$N = 14$
0	$0.02896 - j.12821$	$0.02882 - j.12835$	$0.02881 - j.12836$
1	$0.34364 - j.11979$	$0.34430 - j.12009$	$0.34420 - j.12018$
2	$0.11377 + j.03063$	$0.11312 + j.03084$	$0.11318 + j.03080$
3	$-0.02306 + j.23736$	$-0.02141 + j.23673$	$-0.02137 + j.23679$
4	$-0.12342 + j.02684$	$-0.12317 + j.02713$	$-0.12319 + j.02715$
5	$-0.02539 - j.10213$	$-0.02760 - j.10531$	$-0.02763 - j.10531$
6	$0.05847 - j.01054$	$0.05935 - j.01088$	$0.05934 - j.01089$
7		$0.00745 + j.03266$	$0.00746 + j.03265$
8		$-0.01746 + j.00195$	$-0.01744 + j.00195$
9		$-0.00097 - j.00833$	$-0.00097 - j.00838$
10		$0.00405 - j.00022$	$0.00406 - j.00022$
11			$0.00008 + j.00183$
12			$-0.00082 + j.00002$
13			$0. \quad - j.00035$
14			$0.00015 - j.0$

TABLE III

Chebyshev Expansion Coefficients $\{f_n\}$ for Plane-Wave Scattering by a Parabolic Reflector ($q = 1$, $kw = \pi$)

n	$N = 8$	$N = 12$	$N = 16$
0	$0.15314 + j.10412$	$0.15200 + j.10229$	$0.15199 + j.10228$
2	$0.03078 + j.03082$	$0.02896 + j.03031$	$0.02896 + j.03031$
4	$0.77261 - j.57645$	$0.76771 - j.58310$	$0.76768 - j.58314$
6	$-0.27726 + j.21986$	$-0.28267 + j.22703$	$-0.28269 + j.22706$
8	$-0.14783 + j.04176$	$-0.15246 + j.04607$	$-0.15249 + j.04610$
10		$0.05218 - j.03001$	$0.05235 - j.03001$
12		$0.00975 + j.00026$	$0.00987 + j.00021$
14			$-0.00348 + j.00154$
16			$-0.00032 - j.00012$

Finally, we present the results for a plane wave scattering by a parabolic reflector $\boldsymbol{\rho}(s) = (wt, qwt^2)$. The integrodifferential equation is

$$\int_{-1}^{1} (1 - s^2)^{1/2} f(s) \left\{ k^2 w^2 (1 + 4q^2 t^2) - \frac{\partial^2}{\partial s \, \partial t} \right\} H_0^{(2)}(kw \, |t - s| \, (1 + q(t + s)^2)) = kw(\sin \alpha - 2qt \cos \alpha) \, e^{jkwt(\cos \alpha + qt \sin \alpha)}. \tag{4.4}$$

Results for $q = 1$, $kw = 2\pi$, and $\alpha = \pi/2$ are presented in Table III. The convergence of the coefficients is clearly demonstrated.

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